Simultaneous Effect of Macromixing and Micromixing on Growth Processes

L. T. FAN, B. I. TSAI and L. E. ERICKSON

Kansas State University, Manhattan, Kansas

Interaction of macromixing and micromixing in microbiological flow reactors based on a kinetic model of the Michaelis-Menten type has been treated by considering two general types of flow systems: the system of *n* continuous stirred-tank reactors in series (*n*-CSTR's in series) and the series combination of a plug-flow reactor and a continuous stirred-tank reactor. Systematic schemes to describe the micromixing conditions of a reactor system are presented. Additional micromixing states are considered besides those proposed by Kramers and those proposed by Zwietering if the systems are employed as empirical models of a flow system. When the number of stirred-reactor units in the system of *n*-CSTR's in series is small, micromixing has a significant effect on the growth processes. As the number of reactor units increases, the micromixing effect on the growth processes decreases. The effect of micromixing is also important for a system described by a series combination of a plug-flow reactor and a continuous stirred reactor. In addition, the exit concentrations from the system having the same residence time distribution and the same degree of segregation may be different from each other. Information obtained in this study is pertinent in the design of biological flow reactors and sewage treatment systems.

The significance of mixing in designing fermentors, biological waste treatment systems, and other process systems in which biological growth occurs has begun to be appreciated in recent years. The mixing within continuous flow systems, in general, can be described in terms of two components, macromixing and micromixing (*7* to *14*). The macromixing component specifies the variation in the residence time experienced by molecules flowing through the system. The micromixing component specifies the variation of environment experienced by the molecules during their passage through the system. The effect of residence time distribution (macromixing) on simple growth processes is well known. Many types of continuous reactor systems that can be used for fermentation reaction have been summarized by Herbert (*1*). Reusser (*2*) has discussed the use of a series of completely mixed reactors and the use of a plug-flow reactor for the production of novobiocin. Grieves et al. (*3*) have adapted the mixing model of Cholette and Cloutier (*4*) for modeling the activated sludge process. Bischoff (*5*) has considered optimal design of the continuous fermentation reactor and concluded that a system in which a continuous stirred-tank reactor (CSTR) is followed by a plug-flow reactor (PFR) is the best.

In a previous paper (*6*), the effect of micromixing on microbial growth processes in an isothermal CSTR and in a PFR was considered. It was concluded that the effect of micromixing on a growth process is appreciable and that segregation is unfavorable to the growth process. Since macromixing conditions represented by the CSTR and the PFR systems correspond to idealized or extreme cases, only qualitative observation could be drawn from the previous work concerning the simultaneous effect of macromixing and micromixing on the performance of microbiological flow reactors. In this study interaction of macromixing and micromixing in microbiological flow reactors has been treated by considering two general types of flow systems which represent a wide variety of macromixing states. These flow systems are the system of n continuous stirred-tank reactors (n-CSTR's) in series and the series combination of a continuous stirred-tank reactor and a plug-flow reactor (CSTR-PFR). The total volumes of all the systems considered here are fixed at a constant value, and therefore variation in the system configuration gives rise to a variety of mixing flow models of a reactor system with a fixed volume.

To characterize micromixing, Danckwerts (*10*) has proposed a measure for micromixing, the so-called degree of segregation J, which is defined as the variance of the age between the points in the reactor divided by variance of the age of all molecules in the systems According to this definition

$$J = \frac{\text{var } \alpha_p}{\text{var } \alpha} = \frac{\frac{1}{V}\int_V (\alpha_p - \bar{\alpha})^2 dv}{\int_0^\infty (\alpha - \bar{\alpha})^2 I(\alpha)\, d\alpha}$$

The volume integral $\int_V$ represents the sum over all points. The term point is a volume element small compared with the size of the whole reacting volume but large enough to contain many molecules. A point can be split into parts or combined with other points. Molecules can diffuse into and from it. Therefore the molecules in a point can have a distribution of age and life expectation.

Thus the two extremes of micromixing, complete segregation and maximum mixedness, can be visualized (*7*, *10*). In the former, each fluid element or point is totally isolated or segregated from the other points in the system. In other words all molecules in any one point have the same age; hence variance of ages between points is identical to the variance of ages of all the molecules. Therefore, for any completely segregated system, regardless of its residence time distribution, the degree of segregation J is 1. Of course the PFR is a completely segregated system, and its J is always 1. In the case of maximum mixedness, the points mix at the earliest possible moment. This corresponds to the opposite extreme of segregation. In a continuous stirred-tank system, when micromixing on a molecular scale is attained instantaneously, the variance of age

B. I. Tsai is with Bethlehem Steel Corporation, Homer Research Laboratories, Bethlehem, Pa.

between the points is 0, because all entering fluid will immediately be mixed on the molecular scale with all the fluid still remaining in the tank. J in this case, that is, J in the CSTR with maximum mixedness, is 0. However, it is worth noting that the flow may be completely segregated in the CSTR, in which case J is unity.

It is evident that a CSTR such as a chemostat† may assume any value of J from 0 through 1, but for a system with the residence time distribution other than that of the CSTR, the range of J is less than 1. If the n-CSTR's in series model is used to generate a residence time distribution between the residence time distribution of the CSTR and that of the PFR, the further the residence time distribution of the model deviates from the residence time distribution of the CSTR, the further the J under the state of maximum mixedness deviates positively from 0. It increases up to 1 as the residence time distribution of the model approaches the residence time distribution of the PFR.

It is well known that the residence time distribution alone, in general, is not sufficient to predict uniquely the yield of a chemical reaction. It can however determine the domain of conversion defined by complete segregation and maximum mixedness. The performance equation of a completely segregated flow reactor system regardless of its residence time distribution can be written as (*10*)

$$C_{seg} = \int_0^\infty C_B(t)\, E(t)\, dt \tag{1}$$

Equation (1) implies that the exit concentration from a completely segregated flow reactor system is the expected concentration of all fluid elements over the residence time distribution.

For a reactor system described by the maximum mixedness model, Zwietering (*7*) has obtained the performance equation written as

$$\frac{dC}{d\lambda} = \rho(C) + \frac{E(\lambda)(C - C_o)}{1 - F(\lambda)} \tag{2}$$

with the boundary condition

$$\frac{dC}{d\lambda} = 0 \quad \text{at} \quad \lambda = \infty \tag{2a}$$

The exit concentration from the reactor system is given by the solution of Equations (2) and (2*a*) evaluated at $\lambda = 0$.

Note that for an isothermal linear reaction, the exit concentration from a completely segregated system is identical to the exit concentration from the same reactor under the condition of maximum mixedness regardless of its residence time distribution, that is, the same solution may be obtained by using Equations (1) and (2) under the same operating conditions but under different micromixing conditions (*7*). In addition, for a single CSTR, Equation (2) singularly reduces to the material balance equation at the steady state, namely (*7*)

$$C_o - C - \bar{t}\,\rho(C) = 0 \tag{3}$$

KINETIC MODEL

A Michaelis-Menten form of the rate equation given below is used here to describe the growth kinetics of microorganisms under isothermal conditions (*17*, *18*).

$$\frac{dS}{dt} = -\frac{\mu_{max}\, SX}{Y(K_S + S)} \tag{4}$$

$$\frac{dX}{dt} = \frac{\mu_{max}\, SX}{K_S + S} - k_D X \tag{5}$$

†A chemostat is a continuous fermentor with the residence time distribution of a CSTR.

In dimensionless form, this becomes

$$\frac{dy_1}{d\theta} = -\frac{\tau y_1 y_2}{K_1 + y_1} \tag{6}$$

$$\frac{dy_2}{d\theta} = \tau\left(\frac{y_1 y_2}{K_1 + y_1} - K_2 y_2\right) \tag{7}$$

In these equations μ_{max} is the maximum specific growth rate which occurs when growth is not limited by the substrate or food concentration, K_S is the saturation constant which is equal to the substrate concentration at which the specific growth rate, $\mu_{max}\, S/(K_S + S)$, is one-half the maximum value, and k_D is the specific endogenous microbial attribution rate which is often obtained from the rate that organisms decrease in quantity in a batch system after all of the substrate or food is consumed. In obtaining dimensionless expressions, Equations (6) and (7), from Equations (4) and (5), a normalizing factor $\bar{t}$, which has a dimension of time, is introduced to obtain the dimensionless time θ. The $\bar{t}$ can be the total reacting time in the batch reactor or the mean holding time in the flow reactor. Therefore the quantity $\tau = \mu_{max}\, \bar{t}$ in Equations (6) and (7) can be defined as either the dimensionless maximum specific growth rate or the dimensionless mean holding time in the system. Throughout this work we have defined τ as the dimensionless mean holding time in the overall system.

MACROMIXING AND MICROMIXING STATES

For the n-CSTR's in series model, numerical results have been obtained in this work for the cases of $n = 2$, 5, and 10. More specifically, when $n = 2$, six possible micromixing states have been considered. When $n = 5$ and $n = 10$, only the two extreme states of micromixing have been considered.

The residence time distribution of the n-CSTR's in series model is (*9*, *19*, *20*)

$$E(t) = \frac{n^n}{(n-1)!}\,\frac{t^{n-1}}{(\bar{t})^n}\, e^{-nt/\bar{t}} \tag{8}$$

or in dimensionless form

$$E(\theta) = \frac{n^n}{(n-1)!}\,\theta^{n-1}\, e^{-n\theta} \tag{9}$$

The residence time distribution curves for $n = 1$, 2, 5, 10 are given in Figure 1 (*16*, *20*). Note that as n increases, the spread of its residence time distribution decreases. For any residence time distribution given by the n-CSTR's in series model, there may exist many possible micromixing states lying between the two extremes of complete segregation and maximum mixedness. Probably one of the most convenient ways to represent these intermediate micromixing states is to consider each tank as a subsystem and to

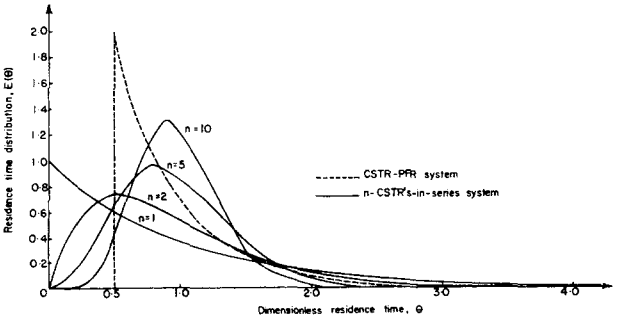

Fig. 1. Dimensionless residence time distribution of n-CSTR's in series system and the CSTR-PFR system (*16*, *20*).

assign an extreme state of micromixing (either complete segregation or maximum mixedness) separately to each tank. This gives rise to a certain number of intermediate micromixing states. For instance, a reactor system may be considered as consisting actually of well-stirred tanks, in each of which mixing occurs on the molecular scale. This is equivalent to stating that the entire reactor system is separated into subsystems (tanks), each of which is under the state of maximum mixedness, but molecular diffusion is not possible between any two adjacent tanks. This condition is termed sequential mixedness (*15*). Note that a system with sequential mixedness is different from that with maximum mixedness. In a system with maximum mixedness infinitely rapid diffusion and association (or grouping) of molecules having the same life expectation (mixing on the molecular scale) are permitted between any parts of the entire system. A reactor system may also be considered as consisting of actually n-CSTR's, each of which is in the state of complete segregation. The state of micromixing corresponding to this condition is termed sequential segregation (*15*). It is worth noting that a system with sequential segregation is different from that with complete segregation. In the former, complete mixing is considered to occur at the junction between two adjacent tanks. In the latter, however, a point is considered to remain segregated from tank to tank throughout the entire system.

Alternatively, a reactor system may be considered as consisting of actually n-CSTR's, some parts of which are in the state of complete segregation or sequential segregation, while others are under the condition of maximum mixedness or sequential mixedness. As the number of CSTR's in the system or the model increases, such combinations will consequently increase. With a larger number of CSTR's in the system, however, the range of possible micromixing states will be reduced appreciably. Such an approach for describing the micromixing states is therefore significant only when there is a small number of CSTR's in the system or in the model.

Equations (1) and (2) or (3) can be applied not only to the entire flow reactor system but also to the subsystems (tanks) within the system whose condition of micromixing is between the two extremes.

In the system of two CSTR's, Zwietering (*7*) has demonstrated that for a second-order reaction, the different micromixing states give rise to different degrees of conversion, and that the degree of conversion decreases with the increase in the extent of micromixing. The extent of micromixing increases in the following order: complete segregation, sequential segregation, sequential mixedness, and maximum mixedness. In the present work two additional micromixing states, sequential segregation-mixedness and sequential mixedness-segregation, have been considered. In the former case, the first CSTR is in the state of complete segregation and the second CSTR in the state of maximum mixedness. In the latter case the sequence is reversed. The degrees of segregation corresponding to different micromixing states for the system of two CSTR's are listed in Table 1. The details of the calculation of degree of segregation are given in Appendix 1.†

For the CSTR-PFR model, a case with the following specific residence time distribution (Figure 1) has been considered in this work:

$$E(\theta) = \begin{cases} 0 & \theta < 0.5 \\ 2\,e^{-2(\theta-0.5)} & \theta \geq 0.5 \end{cases} \qquad (10)$$

This residence time distribution is generated by a reactor model containing a CSTR and a PFR of equal volume in series. This may be either the case of a PFR followed by a CSTR or the case of a CSTR followed by a PFR. Kramers (*8*) considered three possible states of micromixing (cases 1, 4, 5 as shown below) which can be associated with this model. It appears, however, that six possible states of micromixing exist:

1. Complete segregation
2. PFR followed by $(CSTR)_{seg}$
3. $(CSTR)_{seg}$ followed by PFR
4. PFR followed by $(CSTR)_{mm}$
5. $(CSTR)_{mm}$ followed by PFR
6. Maximum mixedness

where the $(CSTR)_{seg}$ represents a CSTR under the state of complete segregation and the $(CSTR)_{mm}$ represents a CSTR under the state of maximum mixedness.

It can be seen that both the macromixing and micromixing states of case 1 are the same as those of case 2 and the mixing state 5 is the same as that of case 6. This may be visualized by actually computing both the degree of segregation of each case and the degree of conversion for a specific reaction under each state of micromixing. Although cases 3 and 4 have the same degree of segregation and the same residence time distribution, the exit concentration from a reactor system represented by case 3 is different from that represented by case 4. This will be discussed in further detail in the last section. Table 2 summarizes the values of J corresponding to cases 1 through 6. The details of calculation are again given in Appendix 1.

TABLE 1. DEGREE OF SEGREGATION OF THE TWO-CSTR'S IN SERIES MODEL UNDER VARIOUS CONDITIONS OF MICROMIXING

Micromixing scale Micromixing states	Degree of segregation J
Complete segregation	1
Sequential segregation	0.7143
Sequential segregation mixedness	0.4286
Sequential mixedness segregation	0.4286
Sequential mixedness	0.1429
Maximum mixedness	0.0275

TABLE 2. DEGREE OF SEGREGATION OF THE PFR-CSTR MODEL UNDER VARIOUS CONDITIONS OF MICROMIXING

Micromixing scale Micromixing states	Degree of segregation J
Complete segregation	1
PFR followed by $(CSTR)_{seg}$	1
$(CSTR)_{seg}$ followed by PFR	0.5471
PFR followed by $(CSTR)_{mm}$	0.5471
$(CSTR)_{mm}$ followed by PFR	0.0942
Maximum mixedness	0.0942

TWO-CSTR's IN SERIES SYSTEM

The yield and conversion of a growth process characterized by a Michaelis-Menten type kinetic equation under the six states of micromixing have been computed:

1. Complete Segregation

The dimensionless exit concentrations of substrate and cell can be obtained by using Equations (1) and (9) as follows:

$$(y_1)_{seg} = \int_0^\infty y_1(\theta)\, 4\theta e^{-2\theta}\, d\theta \qquad (11)$$

† All appendices have been deposited as document No. 01383 with the ASIS National Auxiliary Publications Service, c/o CCM Information Sciences, Inc., 909 Third Ave., New York 10022 and may be obtained for $2.00 for microfiche or $5.00 for photocopies.

$$(y_2)_{seg} = \int_0^\infty y_2(\theta)\, 4\theta e^{-2\theta}\, d\theta \qquad (12)$$

where $y_1(\theta)$ and $y_2(\theta)$ are the batch concentrations obtained from Equations (6) and (7) with the initial or feed concentrations $y_1(0) = y_1^f$ and $y_2(0) = y_2^f$. A numerical procedure, the Runge-Kutta method, has been employed in this work to obtain the desired solution of a set of equations consisting of Equations (6), (7), (11), and (12) (Appendix 2).†

2. Sequential Segregation

The governing equations for the exit concentrations of substrate and cell from the first tank are

$$(y_1)_{seg} = \int_0^\infty y_1(\theta_1)\, e^{-\theta_1}\, d\theta_1 \qquad (13)$$

$$(y_2)_{seg} = \int_0^\infty y_2(\theta_1)\, e^{-\theta_1}\, d\theta_1 \qquad (14)$$

where $y_1(\theta_1)$ and $y_2(\theta_1)$ are the solutions obtained from Equations (6) and (7) with θ replaced by θ_1, which is $2t/\bar{t}$, and with τ replaced by $\tau/2$.

The exit concentrations from the second tank can be obtained in the same manner but with initial conditions given by

$$y_1(0) = (y_1)_{seg} \text{ from the exit of the first tank}$$
$$y_2(0) = (y_2)_{seg} \text{ from the exit of the first tank}$$

Note that there is complete mixing at the junction between the two tanks and that the effluent of the first tank naturally becomes the feed of the second one.

3. Sequential Segregation Mixedness

The exit concentrations from the first CSTR with complete segregation can be obtained by using Equations (13) and (14), and they become the feed concentrations to the second CSTR with maximum mixedness. According to Equations (3), (6), and (7), the material balance equations of the substrate and the cell concentrations for the second CSTR are

$$y_1^f - y_1 - \frac{\tau y_1 y_2}{2(K_1 + y_1)} = 0 \qquad (15)$$

$$y_2^f - y_2 + \frac{\tau}{2}\left(\frac{y_1 y_2}{K_1 + y_1} - K_2 y_2\right) = 0 \qquad (16)$$

The exit concentrations from the second tank are obtained by solving these equations for y_1 and y_2 (see Appendix 3).†

4. Sequential Mixedness Segregation

In this case, the sequence of the reactor arrangement is reversed from that of the preceding case. The exit concentrations from the first CSTR with maximum mixedness can be obtained by using Equations (15) and (16) and they become the feed concentrations to the second CSTR with complete segregation. The exit concentrations from the second tank can be obtained from Equations (13) and (14). The initial conditions for solving $y_1(\theta)$ and $y_2(\theta)$ in these equations are

$$y_1(0) = (y_1)_{mm} \text{ from the exit of the first tank}$$
$$y_2(0) = (y_2)_{mm} \text{ from the exit of the first tank}$$

5. Sequential Mixedness

Equations (15) and (16) are applicable to both the first tank and the second tank. The exit concentrations from the first tank become the feed concentrations to the second tank.

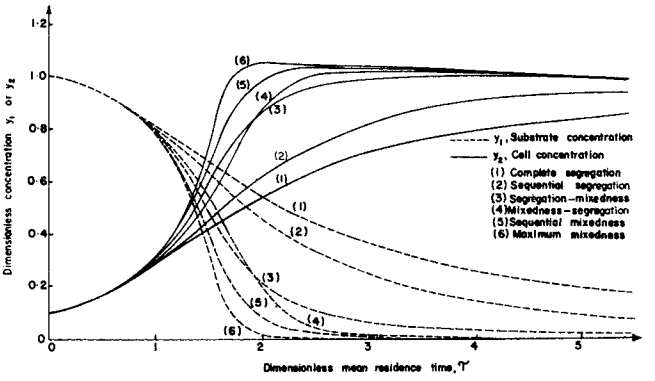

Fig. 2. Exit concentration of substrate and cell from the system of two CSTR's in series.

6. Maximum Mixedness

According to Equations (2) and (2a) the exit concentrations of substrate and cell can be obtained by solving the system of equations

$$\frac{dy_1}{d\lambda^*} = \frac{\tau y_1 y_2}{K_1 + y_1} + \frac{E(\lambda^*)}{1 - F(\lambda^*)}\,[y_1 - y_1^f] \qquad (17)$$

$$\frac{dy_2}{d\lambda^*} = -\frac{\tau y_1 y_2}{K_1 + y_1} + K_2 y_2 + \frac{E(\lambda^*)}{1 - F(\lambda^*)}\,[y_2 - y_2^f] \qquad (18)$$

subject to boundary conditions

$$\frac{dy_1}{d\lambda^*} = 0 \quad \text{at} \quad \lambda^* = \infty \qquad (19)$$

$$\frac{dy_2}{d\lambda^*} = 0 \quad \text{at} \quad \lambda^* = \infty \qquad (20)$$

where

$$\lambda^* = \frac{\lambda}{\bar{t}} \qquad (21)$$

$$E(\lambda^*) = 4\lambda^*\, e^{-2\lambda^*} \qquad (22)$$

$$F(\lambda^*) = 1 - (2\lambda^* + 1)\, e^{-2\lambda^*} \qquad (23)$$

The exit concentrations are the solution of Equations (17) through (23) evaluated at $\lambda = 0$.

The numerical results of all six cases are given in Figure 2.

n-CSTR's IN SERIES SYSTEM

1. Complete Segregation

The exit concentrations of substrate and cell can be obtained by using Equations (1) and (9) as follows:

$$(y_1)_{seg} = \int_0^\infty y_1(\theta)\, \frac{n^n}{(n-1)!}\, \theta^{n-1}\, e^{-n\theta}\, d\theta \qquad (24)$$

$$(y_2)_{seg} = \int_0^\infty y_2(\theta)\, \frac{n^n}{(n-1)!}\, \theta^{n-1}\, e^{-n\theta}\, d\theta \qquad (25)$$

where $y_1(\theta)$ and $y_2(\theta)$ are the batch concentrations obtained from Equations (6) and (7) with the initial concentrations $y_1(0) = y_1^f$ and $y_2(0) = y_2^f$.

2. Maximum Mixedness

The exit concentrations can be obtained from Equations (17) through (21) with

$$E(\lambda^*) = \frac{n^n}{(n-1)!}\, (\lambda^*)^{n-1}\, e^{-n\lambda^*} \qquad (26)$$

†See footnote on page 691.

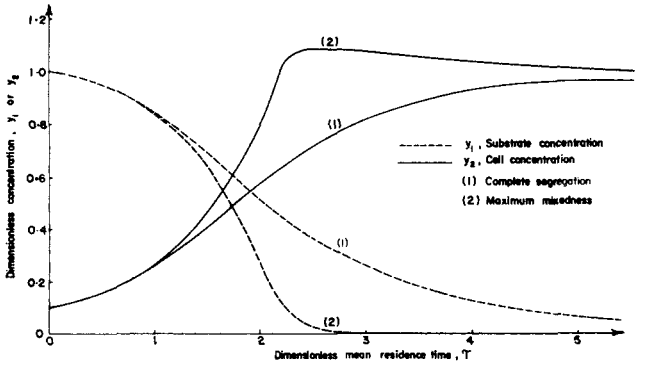

Fig. 3. Exit concentration of substrate and cell from the system of five CSTR's in series.

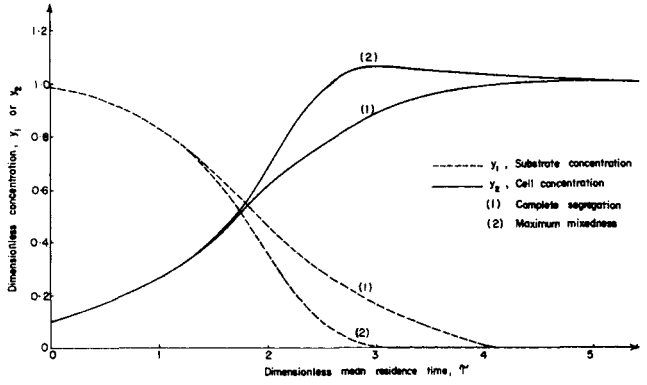

Fig. 4. Exit concentration of substrate and cell from the system of ten CSTR's in series.

$$F(\lambda^*) = 1 - e^{-n\lambda^*}\left[\sum_{j=1}^{n}\frac{(n\lambda^*)^{j-1}}{(j-1)!}\right] \tag{27}$$

As stated previously, the systems of $n = 5$ and $n = 10$ have been considered in the present work.

The exit concentration of substrate and cell from the system of five CSTR's in series under two extreme states of micromixing is shown in Figure 3. The results for the system of ten CSTR's in series are shown in Figure 4.

THE CSTR-PFR SYSTEM

1. Complete Segregation

The exit concentrations of substrate and cell from the system with complete segregation can be obtained by use of Equations (1), (6), (7), and (10) as follows:

$$(y_1)_{seg} = \int_{0.5}^{\infty} y_1(\theta)\, 2e^{-2(\theta-0.5)}\, d\theta \tag{28}$$

$$(y_2)_{seg} = \int_{0.5}^{\infty} y_2(\theta)\, 2e^{-2(\theta-0.5)}\, d\theta \tag{29}$$

The details of calculation are given in Appendix 4.†

2. PFR Followed by $(CSTR)_{seg}$

As mentioned in the introductory section, the PFR may be considered to be always in the state of complete segregation and its residence time distribution is a delta function, that is

$$E(\theta_1) = \delta(\theta_1 - 1) \tag{30}$$

The exit concentration of substrate from the PFR may then be expressed as

$$(y_1)_{PFR} = \int_0^{\infty} y_1(\theta_1)\, E(\theta_1)\, d\theta_1$$

$$= \int_0^{\infty} y_1(\theta_1)\, \delta(\theta_1 - 1)\, d\theta_1$$

$$= y_1(1) \tag{31}$$

and similarly the corresponding cell concentration can be expressed as

$$(y_2)_{PFR} = y_2(1) \tag{32}$$

These values of $(y_1)_{PFR}$ and $(y_2)_{PFR}$ are found by simultaneously integrating Equations (6) and (7) subject to the initial conditions $y_1(0) = y_1^f$ and $y_2(0) = y_2^f$ and then evaluating the resulting expressions of $y_1(\theta_1)$ and $y_2(\theta_1)$ at $\theta_1 = 1$. For the second tank with complete segregation we have

$$(y_1)_{seg} = \int_0^{\infty} y_1(\theta_1) e^{-\theta_1}\, d\theta_1 \tag{33}$$

$$(y_2)_{seg} = \int_0^{\infty} y_2(\theta_1) e^{-\theta_1}\, d\theta_1 \tag{34}$$

where $y_1(\theta_1)$ and $y_2(\theta_1)$ are obtained by integrating Equations (6) and (7) subject to initial conditions

$$y_1(0) = (y_1)_{PFR} \tag{35}$$

$$y_2(0) = (y_2)_{PFR} \tag{36}$$

It can be shown mathematically that the $(y_1)_{seg}$ and $(y_2)_{seg}$ obtained from Equations (33) and (34) for the present system of a PFR followed by a $(CSTR)_{seg}$ are identical with those obtained from Equations (28) and (29) for the completely segregated system (see Appendix 4). This can also be visualized from the fact that all the exit fluid particles or points from the PFR have the same age and all the points remain completely segregated before they enter the completely segregated CSTR in both cases. [Although there is complete mixing at the junction of two vessels in the present system of a PFR followed by $(CSTR)_{seg}$, it does not influence the micromixing state of the fluid since all the points mixed at the junction contain molecules of the same age and the condition of complete segregation remains fulfilled.] This is not the case for the system of a $(CSTR)_{seg}$ followed by a PFR, which will be considered next.

3. $(CSTR)_{seg}$ Followed by PFR

In this case the sequence of the reactor arrangement is reversed from that of the preceding case. The use of Equations (33) and (34) gives rise to the exit concentrations from the first tank or the feed concentrations to the second vessel. However, the initial conditions for $y_1(\theta_1)$ and $y_2(\theta_1)$ are

$$y_1(0) = y_1^f$$

$$y_2(0) = y_2^f$$

The exit concentrations from the second vessel are obtained by properly using Equations (31) and (32).

Since complete mixing occurs at the junction of the two vessels, or more specifically at the exit of the $(CSTR)_{seg}$, which is also the entrance to the PFR, any point in the PFR contains the molecules whose ages are distributed according to the exponential decay function. In spite of the existence of the plug flow in the second vessel, a point in the PFR no longer contains molecules of the same age. This system, that is, the system of a $(CSTR)_{seg}$ followed by a PFR, can no longer be considered as being completely segregated.

†See footnote on page 691.

4. PFR followed by $(CSTR)_{mm}$

The exit concentrations from the PFR can be obtained from Equations (31) and (32), which become the feed concentrations to the $(CSTR)_{mm}$. The exit concentrations from the $(CSTR)_{mm}$ are obtained from Equations (15) and (16).

5. $(CSTR)_{mm}$ followed by PFR

In this case the sequence of the reactor arrangement is reversed from that of the preceding case. The use of Equations (15), (16), (31), and (32) leads to the desired solution. It can be shown that the exit concentrations of substrate and cell from the reactor system are identical with those from the system with maximum mixedness, which will be considered next (see Appendix 5).* Evidently, in this case the condition of maximum mixedness—"mixing as early as possible"—is fulfilled.

6. Maximum Mixedness

The exit concentrations of substrate and cell can be obtained by using Equations (17) through (21). However the cumulative residence time distribution in this case is

$$F(\theta) = \begin{cases} 0 & \theta \leq 0.5 \\ 1 - e^{-2(\theta - 0.5)}, & \theta \geq 0.5 \end{cases} \qquad (37)$$

The details of calculation are shown in Appendix 5. The numerical results of all six cases are shown in Figure 5.

DISCUSSION OF RESULTS AND CONCLUSION

Values of the parameters employed in constructing Figure 2 for the system of two reactors in series are $y_2^f = 0.1$, $y_1^f = 1.0$, $K_1 = 0.1$, $K_2 = 0.02$, in all cases. Figure 2 shows that the exit substrate concentration for the case of complete segregation [dotted line (1)], for which $J = 1$, is always higher than those of other cases at any dimensionless time τ, especially when τ is greater than 1. On the other hand, the exit substrate concentration corresponding to the case of maximum mixedness [dotted line (6)], for which $J = 0.0275$, is always lower than those of other cases. The exit substrate concentration corresponding to sequential segregation [dotted line (2)], for which $J = 0.7143$, is slightly lower than that of complete segregation and the exit concentration corresponding to the case of sequential mixedness [dotted line (5)] for which $J = 0.1429$ is slightly higher than that of maximum mixedness. Dotted lines (3) and (4) are the substrate concentrations corresponding to the case of sequential segregation mixedness and the case of sequential mixedness segregation, respectively. For τ less than approximately 1.9, line (3) is higher than line (4) and for larger τ the relationship is reversed. It is worth noting that lines (3) and (4) do not coincide even though both correspond to the same degree of segregation of $J = 0.4286$. This appears to provide a concrete numerical example to support Rippin's contention (*14*) that reactors having the same residence time distribution and the same degree of segregation may nevertheless produce different degrees of conversion.

Figure 2 also shows the corresponding cell concentrations by solid lines. (Note that the maximum value of y_2 may be greater than 1, since $y_1^f = 1.0$ and $y_2^f = 0.1$.) It indicates that the higher degree of substrate conversion leads to higher cell concentrations for this model and the particular parameter values employed. In general, it appears that segregation is unfavorable to substrate conversion and the growth of microorganisms. Because the endogeneous term is included in the kinetic model for cell growth, the higher degree of substrate conversion may not always imply a higher cell concentration especially at a higher τ. If the endogeneous term is not included in the kinetic model for cell growth, the higher degree of substrate conversion always gives rise to a higher cell concentration. In fact, the decrease in substrate concentration is exactly proportional to the increase in the cell concentration.

As mentioned previously, for the n-CSTR's in series model, as the number of tanks n increases, the corresponding residence time distribution generated by the model approaches to the residence time distribution of the PFR. The system having such a residence time distribution approaches the state of segregation. Thus as the number of tanks increases, the micromixing effect on the reaction yield subsequently becomes negligible. This can be seen from Figures 2 through 4. In Figures 2, 3, and 4 substrate and cell concentrations are shown for the systems of n = 2, 5, 10, respectively, under the two extreme states of micromixing. It can be seen that the micromixing effect on both substrate and cell concentrations in the two-tank systems is more appreciable than that in the five-tank system, which in turn, is more appreciable than that in the ten-tank system.

Figure 5 shows that for the CSTR-PFR system, the case of complete segregation and the case of a PFR followed by a $(CSTR)_{seg}$ give rise to the same degree of segregation of $J = 1$ and the same exit concentration of substrate [dotted lines (1) and (2) overlapping], which is higher than those of other cases for the same τ. However, the substrate concentration corresponding to the case of maximum mixedness and that for the case of a $(CSTR)_{mm}$ followed by a PFR [dotted lines (6) and (5) coincide], for which $J = .0942$, are always lower than those for other cases. Dotted lines (3) and (4) are the substrate concentrations corresponding to the cases of a PFR followed by a $(CSTR)_{seg}$ and a $(CSTR)_{seg}$ followed by a PFR, respectively. For a τ less than about 2.5, line (4) is higher than line (3) and for a larger τ this trend is reversed. It is worth noting again that lines (3) and (4) do not coincide even though they correspond to the same residence time distribution and the same degree of segregation of $J = 0.5471$.

Figure 5 also shows that, in general, the higher substrate conversion leads to the higher cell concentration. As τ increases, however, the endogeneous respiration of the cell becomes predominate, especially for the systems with higher degrees of micromixing. It appears that the micromixing favors not only the consumption of the substrate, but also the consumption of the cell by endogeneous respiration. It is worth noting that the exit cell concentration from the $(CSTR)_{mm}$ + PFR system exceeds those from the other system when τ is not large. This is in agreement with Bischoff's optimization results (5) in which he found that the $(CSTR)_{mm}$ + PFR system gives the optimal fermentation system design.

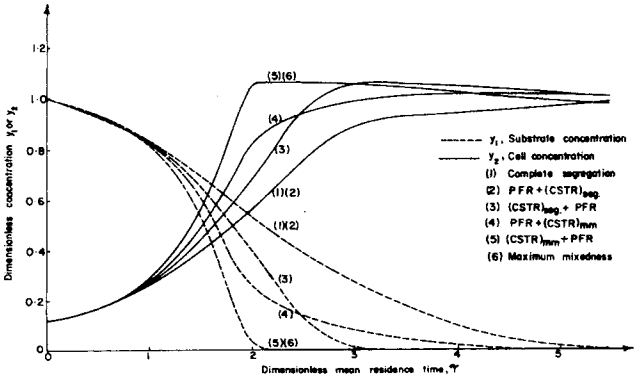

Fig. 5. Exit concentration of substrate and cell from the PFR-CSTR system.

†See footnote on page 691.

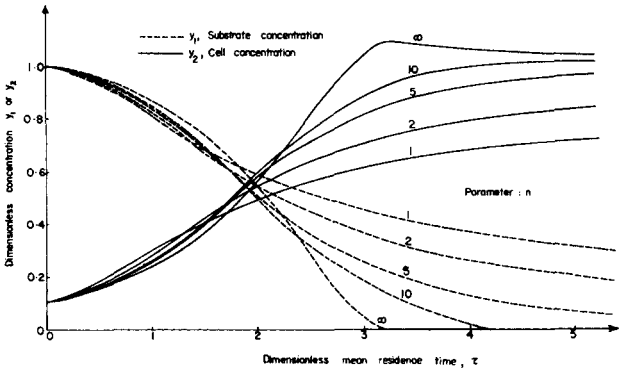

Fig. 6. Exit concentration of substrate and cell from the system of n-CSTR's in series with complete segregation.

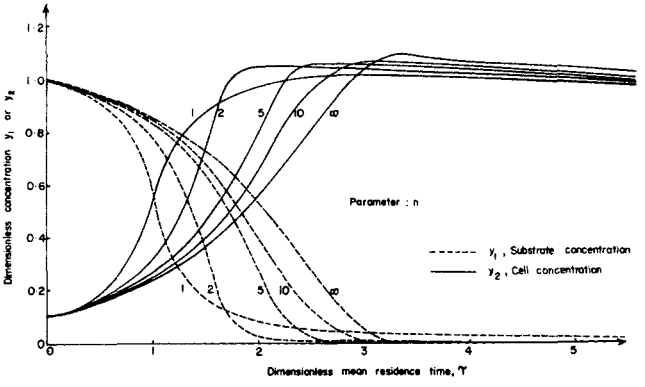

Fig. 7. Exit concentration of substrate and cell from the system of n-CSTR's in series with maximum mixedness.

Figures 6 and 7 show the effect of the number of reactors in the n-CSTR's in series system on the exit concentrations of substrate and cell from the reactor systems. The exit concentrations of substrate and cell from the n-CSTR's systems approach to those from the PFR for both extreme cases of micromixing, that is, complete segregation and maximum mixedness. Figure 6 shows the exit concentrations of substrate and cell from the n-CSTR's systems with $n = 1, 2, 5, 10, \infty$, under the condition of complete segregation. For a small τ, the exit concentrations of substrate (dotted lines) increase with n. For a larger τ, however, the exit concentrations of substrate decrease appreciably with n. With $n = 10$, the exit substrate concentration from the reactor system is quite close to that from the PFR. Figure 7 shows the exit concentrations of substrate and cell from the n-CSTR's systems under the condition of maximum mixedness. For a small τ, the exit concentrations of substrate (dotted lines) substantially increase with n. For a larger τ, however, this trend is reversed. Solid lines in Figures 6 and 7 indicate that for a small τ, a small number of CSTR's corresponding to a high degree of macromixing is favorable to the growth of cells. For a large τ, however, this trend is reversed.

From the above analysis and discussions, several general conclusions about the n-CSTR's in series system may be drawn.

1. When the number of CSTR's in the n-CSTR's in series system or model is small, the micromixing effect on the growth processes is very appreciable and is as important as the macromixing effect.

2. When the number of CSTR's becomes large ($n \geq 10$), the micromixing effect on the growth processes becomes negligibly small.

3. In general for any residence time distribution the segregation effect is unfavorable to the conversion of the substrate or to the growth processes.

4. The exit concentrations from the systems having the same residence time distribution and the same degree of segregation may be different from each other.

Simultaneous effects of macromixing and micromixing on the growth processes are treated extensively in this study. Information obtained in this study is pertinent in the design of biological flow reactors and sewage treatment systems.

ACKNOWLEDGMENT

This work was partly supported by the U.S. Department of Interior, Federal Water Pollution Control Administration (Projects WP-01141-01, WP-01141-02, and 17090 ELL) and by Public Health Service Research Career Development Award No. 1 K4 GM35397-01.

NOTATION

C_o = concentration of reactant in the feed stream, M/L^3t
C = concentration of reactant in the reactor, M/L^3t
$C(t)$ = concentration predicted by the batch kinetics, M/L^3t
C_{seg} = average exit concentration from a completely segregated system, M/L^3t
$E(t)$ = exit age or residence time distribution function, t^{-1}
F = cumulative life expectation distribution, dimensionless
$I(\alpha)$ = internal age distribution of molecules in the system, t^{-1}
$I_p(\alpha)$ = age distribution of molecules within the point, t^{-1}
J = degree of segregation, dimensionless
K_S = saturation constant, M/L^3
$K_1 = K_S/S_o$ = saturation constant, dimensionless
$K_2 = k_D/\mu_{max}$ = endogeneous metabolism constant, dimensionless
k_D = endogeneous metabolism constant, t^{-1}
n = number of tanks
q = flow rate, L^3/t
S = substrate concentration, M/L^3
S_o = influent concentration of the substrate, M/L^3
t = time, t
$\bar{t} = V/q$ = mean holding time of the system, t
V = total volume of the reactor system, L^3
X = cell concentration, M/L^3
Y = yield factor, dimensionless
$y_1 = S/S_o$ = substrate concentration, dimensionless
$y_2 = X/YS_o$ = cell concentration, dimensionless
y_1^f = feed concentration of substrate, dimensionless
y_2^f = feed concentration of cell, dimensionless
$(y_1)_{mm}$ = exit substrate concentration from a maximum mixedness system, dimensionless
$(y_2)_{mm}$ = exit cell concentration from a maximum mixedness ness system, dimensionless

Greek Letters

α = age of a molecule, t
α^* = age of a molecule, dimensionless
$\bar{\alpha}$ = mean age of molecules in the system, t
α_p = mean age of molecules within a point in the system, t
α_2 = age counted from the entrance of the second vessel, t
λ = life expectation of a molecule, t
λ^* = life expectation, dimensionless
θ = time based on overall system, $t/\bar{t}$ dimensionless
θ_1 = time based on each subsystem, dimensionless
$\rho(c)$ = batch kinetic expression for the reaction rate, M/L^3t

$\xi(\lambda)$ = average age of the molecules having a life expectation, t

μ_{max} = maximum specific growth rate, t

$\tau = \mu_{max}\bar{t}$ = mean holding time in the overall system, dimensionless

LITERATURE CITED

1. Herbert, D., *Soc. Chem. Ind. Monograph No. 12*, London, 21 (1961).
2. Reusser, F., *Appl. Microbiol.*, **9**, 361 (1961).
3. Grieves, R. B., W. F. Milbury, and W. O. Pipes, *J. Water Pollution Fed.*, **36**, 619 (1964).
4. Cholette, A., and L. Cloutier, *Can. J. Chem. Eng.*, **37**, 105 (1959).
5. Bischoff, K. B., *ibid.*, **44**, 281 (1966).
6. Tsai, B. I., L. E. Erickson, and L. T. Fan, *Biotech. Bioeng.*, **11**, 181 (1969).
7. Zwietering, Tr. N., *Chem. Eng. Sci.*, **11**, 1 (1959).
8. Kramers, H., *Chem. Eng. Sci.*, **8**, 45 (1958).
9. Danckwerts, P. V., *ibid.*, **2**, 1 (1953).
10. *Ibid.*, **8**, 93 (1958).
11. Douglas, J. M., *Chem. Eng. Progr. Symp. Ser. No. 48*, **60**,1 (1964).
12. Ng, D. Y. C., Ph. D. thesis, Imperial Coll., Univ. London (1965).
13. Weinstein, H., and R. J. Adler, *Chem. Eng. Sci.*, **22**, 65 (1967).
14. Rippin, D. W. T., *ibid.*, **22**, 247 (1967).
15. Novosad, Z., and J. Thyn, *Collection Czech. Chem. Commun.*, **31**, 3710 (1966).
16. Himmelblau, D. M., and K. B. Bischoff, "Process Analysis and Simulation," Wiley, New York (1968).
17. Michaelis, L., and M. Menten, *Biochem. Z.*, **49**, 333 (1913).
18. Monod, J., *Ann. Rev. Microbiol.*, **3**, 371 (1949).
19. Ham, A., and H. S. Coe, *Chem. Met. Eng.*, **19**, 663 (1918).
20. MacMullin, R. B., and M. Weber, *Trans. AIChE*, **31**, 409 (1935).

Manuscript received July 9, 1969; revision received August 29, 1969; paper accepted September 2, 1969. Paper presented at AIChE Atlanta meeting.

The Calculation of Unsteady State Multicomponent Distillation Using Partial Differential Equations

ADAM OSBORNE

Osborne & Associates, Berkeley, California

A method is presented for calculating unsteady state multicomponent distillation using partial differential equations. The equations are solved by treating them as continuous in theoretical stage direction and stepped in time. Although some restrictions were placed on the problems actually solved, the method itself is not restricted by number of trays, number of components, number, type or combination of upsets, or by the thermodynamics of the system, and the method readily lends itself to hybrid computer solution.

There are two parts to the development of a mathematical simulation of unsteady state multicomponent distillation. First it is necessary to consider the physical and chemical properties of the system, by which to formulate its mathematical model. Second we must find a way of solving the equations of the mathematical formulation.

Marshall and Pigford (*8*) first published the differential equations describing the transient behavior of one distillation plate. Their work, and other early work was confined to binary systems, or to systems in which an additional nonvolatile component was present. A number of assumptions were made by Marshall and Pigford (*8*), and became standard in later work:

1. The equations were considered continuous in time, and differenced in theoretical stage. This would appear logical.

2. Linear equilibrium relationships between vapor and liquid compositions were assumed, of the form

$$y = mx + b \tag{1}$$

3. Constant molal overflow was assumed. Later workers assumed that liquid flow rate was a linear function of temperature profile.

4. Ideal trays were assumed.

Authors who followed Marshall and Pigford include Lapidus and Amundson (*6*), Acrivos and Amundson (*1*), Mickley, Sherwood, and Reed (*9*), Rose and coworkers (*11* to *15*), Rosenbrock (*16*), Huckaba and Tour (*5*), Davidson (*2*), Wilkinson and Armstrong (*20*), Waggoner (*19*), and Holland (*4*).

The limiting assumptions of Marshall and Pigford (*8*)